Computable Jordan Decomposition of Linear Continuous Functionals on $\mathbf{C[0;1]}$

Tehereh Jafarikhah, Massoud Pourmahdian, Klaus Weihrauch

University of Tarbiat Modares, Tehran
Amirkabir University of Technology, Tehran
University of Hagen, Hagen,

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Weak decomposition theorems

Linear continuous functionals $F : C[0;1] \to \mathbb{R}$:

Theorem For every F there are non-decreasing functionals F^+, F^- such that $F = F^+ - F^-$.

Functions of bounded variation $g : [0;1] \to \mathbb{R}$

Theorem For every g there are non-decreasing functions g^+, g^- such that $g = g^+ - g^-$

Signed measures on $[0;1]$ of finite variation norm $\mu : \mathcal{B} \to \mathbb{R}$

Theorem For every μ there are non-negative measures μ^+, μ^- such that $\mu = \mu^+ - \mu^-$.

In each case there is a minimal decomposition.

linear continuous functionals

$C[0;1]$:= the set of continuous functions $h : [0;1] \to \mathbb{R}$

$\|h\| := \max_{|x| \leq 1} |h(x)|$

$C'[0;1]$:= the set of linear continuous functions $F : C[0;1] \to \mathbb{R}$

$\|F\| := \sup_{\|h\| \leq 1} |F(h)|$

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Examples

– $F(h) := \int h(x)\,\mathrm{d}x$ (Riemann integral)

– $F(h) := -h(1/2)$

bv-functions and Riemann-Stieltjes integral

For $g : [0;1] \to \mathbb{R}$ and partition
$Z = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$,

$S(g, Z) := \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$ (sum of heights of jumps)
$\mathrm{Var}(g) := \sup_Z S(g, Z)$ (Variation of g)

For continuous $h : [0;1] \to \mathbb{R}$
$S(g, h, Z) := \sum_{i=1}^{n} h(x_i)(g(x_i) - g(x_{i-1}))$
$\int h \, \mathrm{d}g := \lim_Z S(g, h, Z)$ (Riemann-Stieltjes integral)
$\int h \, \mathrm{d}g$ exists if $\mathrm{Var}(g) < \infty$

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Normalization: For every g of bounded variation there is some g' of bounded variation such that for $0 < x < 1$,
$g'(0) = 0$, $\lim_{y \nearrow x} g(y) = g(x)$ and $\int h \, \mathrm{d}g = \int h \, \mathrm{d}g'$.

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Examples

– $g(x) := x$, $\int h \,\mathrm{d}g = \int h(x) \,\mathrm{d}x$ (Riemann integral)
– $g(x) :=$ (0 if $x \leq 1/2$, -1 else). $\int h \,\mathrm{d}g = -h(1/2)$

signed measures and integral of cont. functions

$\mu : \mathcal{B} \to \mathbb{R}$ $\quad \mathcal{B} =$ the Borel-subsets of $[0;1]$

$\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i)$ for pairwise disjoint A_i.

For partition Π of $[0;1]$ into intervals (a,b), $[a;b)$ etc.

$S(\mu, \Pi) := \sum_{I \in \Pi} |\mu(I)|$

$\|\mu\| := \sup_{\Pi} S(\mu, \Pi)$ (Variation norm)

For continuous $h : [0;1] \to \mathbb{R}$,

$S(\mu, h, \Pi) := \sum_{I \in \Pi} \inf h[I] \cdot \mu(I)$

$\int h \, \mathrm{d}\mu := \lim_{\Pi} S(\mu, h, \Pi)$

$\int h \, \mathrm{d}\mu$ exists if $\|\mu\| < \infty$

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Examples

– Lebesgue measure λ, $\lambda(a;b) := b-a$, $\int h \, \mathrm{d}\lambda = \int h(x) \, \mathrm{d}x$

– $\mu(I) := (-1 \text{ if } 1/2 \in I, \quad 0 \text{ else})$, $\int h \, \mathrm{d}\mu : -h(1/2)$

Riesz representation theorem

Theorem

1. For every linear continuous functional F there is a unique normalized function g of bounded variation such that
   $F(h) = \int h \, \mathrm{d}g$ and $\|F\| = \mathrm{Var}(g)$.
2. For every normalized function g of bounded variation there is a unique signed measure μ such that
   $\int h \, \mathrm{d}g = \int h \, \mathrm{d}\mu$ and $\mathrm{Var}(g) = \|\mu\|$.
3. for every signed measure μ there is a unique linear continuous functional F such that
   $\int h \, \mathrm{d}\mu = F(h)$ and $\|\mu\| = \|F\|$.

The first statement: Riesz representation theorem

Strong Jordan decomposition

Call $Y = Y^+ - Y^-$ a minimal decomposition iff
$Y = Z^+ - Z^-$ implies $Y^+ \leq Z^+$ and $Y^- \leq Z^-$.

Theorem (Jordan decomposition)

1. Every linear continuous functional F has a minimal decomposition $F = F^+ - F^-$ into non-decreasing linear continuous functionals.
2. Every normalized function g of bounded variation has a minimal decomposition $g = g^+ - g^-$ into non-decreasing normalized functions of bounded variation.
3. Every signed measure μ has a minimal decomposition $\mu = \mu^+ - \mu^-$ into non-negative measures.

Theorem

1. A decomposition $F = H^+ - H^-$ into non-negative functionals is minimal iff $\|F\| = \|H^+\| + \|H^-\|$.
2. A decomposition $g = s^+ - s^-$ into non-decreasing functions is minimal iff $\mathrm{Var}(g) = \mathrm{Var}(s^+) + \mathrm{Var}(s^-)$.
3. A decomposition $\mu = \nu^+ - \nu^-$ into non-negative measures is minimal iff $\|\mu\| = \|\nu^+\| + \|\nu^-\|$.

simple computability on operators and bv-functions

- $C[0;1]$: Cauchy representation δ_C, rational polygons are dense
- $C'[0;1]$: $[\delta_C \to \rho]$ (represents all continuous functions)

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– $C[0;1]$: Cauchy representation δ_C, rational polygons are dense

– $C'[0;1]$: $[\delta_C \to \rho]$ (represents all continuous functions)

– $\mathrm{BV}_0 :=$ normalized functions of bounded variation:
($\int h \, \mathrm{d}g$ can be computed from $\mathrm{Var}(g)$ and g restricted to a countable dense subset.)

$\delta_0(p) = g$ iff
p encodes a sequence $(x_i, y_i)_{i \in \mathbb{N}}$ from $\mathbb{R}^2$ such that
– the set $A_p := \{x_i \mid i > 1\}$ is dense and g is continuous on A_p
– $(\forall i)\, g(x_i) = y_i$, and
– $x_0 = y_0 = 0$ and $x_1 = 1$

simple computability on measures

– BM the bounded non-negative Borel measures on $[0;1]$:

$\delta_m\langle p, q\rangle = \mu$ iff $\mu[0;1] = \rho(p)$ and q is a list of all (a, I) ($a \in \mathbb{Q}$, I open rational interval) such that $a < \mu(I)$.

Lemma

δ_m is the greatest (poorest) representation γ of BM such that $(h, \mu) \mapsto \int h \,\mathrm{d}\mu$ is (δ_C, γ)-computable.

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Lemma
δ_m is the greatest (poorest) representation γ of BM such that $(h, \mu) \mapsto \int h \, \mathrm{d}\mu$ is (δ_C, γ)-computable.

– SBM the set of signed Borel measures:
$\delta_{\mathrm{sm}}\langle p, q\rangle = \delta_m(p) - \delta_m(q)$ (difference of non-negative measures)

Computable Riesz representation

Theorem

1. The function $(F, \|F\|) \mapsto (g, \mathrm{Var}(g))$ is computable where $F(h) = \int h \,\mathrm{d}g$ and $\|F\| = \mathrm{Var}(g)$.
2. The function $(g, \mathrm{Var}(g)) \mapsto (\mu, \|\mu\|)$ is computable where $\int h \,\mathrm{d}g = \int h \,\mathrm{d}\mu$ and $\mathrm{Var}(g) = \|\mu\|$.
3. The function $(\mu, \|\mu\|) \mapsto (F, \|F\|)$ is computable where $\int h \,\mathrm{d}\mu = F(h)$ and $\|F\| = \|\mu\|$.

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1. is the computable Riesz representation theorem.

Corollary The three representations of the space $C'[0;1]$ via the names of $(F, \|F\|)$ of $(g, \mathrm{Var}(g))$ and of $(\mu, \|\mu\|)$ are equivalent. (correspondingly for the space of normalized bv-functions and of the signed measures)

Computable Jordan decomposition

Theorem

The following functions for minimal decomposition and their inverses are computable:

$$
\begin{aligned}
(F, \|F\|) &\mapsto (F^+, F^-) \\
(g, \mathrm{Var}(g)) &\mapsto (g^+, g^-) \\
(\mu, \|\mu\|) &\mapsto (\mu^+, \mu^-)
\end{aligned}
$$

Computable Jordan decomposition

Theorem
The following functions for minimal decomposition and their inverses are computable:

$$\begin{array}{rcl}(F, \|F\|) & \mapsto & (F^+, F^-) \\ (g, \mathrm{Var}(g)) & \mapsto & (g^+, g^-) \\ (\mu, \|\mu\|) & \mapsto & (\mu^+, \mu^-)\end{array}$$

Corollary The representations of $C'[0;1]$ via the names of the minimal decompositions (F^+, F^-), (g^+, g^-) and (μ^+, μ^-) are equivalent to the former ones.
(correspondingly for the space of normalized bv-functions and of the signed measures)

exceptional points

For interval I let $\|F\|_I := \sup\{|F(h)| \mid \mathrm{supp}(h) \subseteq I, \ \ \|h\| \leq 1\}$,

For functionals F, bv-functions g and signed measures μ such that $F(h) = \int h \, \mathrm{d}g = \int h \, \mathrm{d}\mu$

$$\lim_{x \in I, \mathrm{lg(I)} \to 0} \|F\|_I = |\lim_{y \searrow x} g(y) - \lim_{y \nearrow x} g(y)| = |\mu(\{x\})|$$

The three concepts are equivalent:

$$\begin{aligned} x \text{ contributes to } F \quad &:\Longleftrightarrow \quad \|F\|_x := \inf_{x \in I} \|F\|_I > 0\,, \\ g \text{ is discontinuous at } x \quad &\Longleftrightarrow \quad \lim_{y \searrow x} g(y) \neq g(x)\,, \\ x \text{ contributes to } \mu \quad &:\Longleftrightarrow \quad \mu(\{x\}) \neq 0\,. \end{aligned}$$

Our names do not tell us where these exceptional points are.